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On the Surfaces with Plane or Spherical Curves of Curvature.

(For first part see vol. XI, No. 1.)

BY PROF. CAYLEY.

I consider next the case

$PS4^0 = \text{Serret's third case of } PS.$

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ ax + by &= lm, \\ Aa + Bb &= l, \\ x^2 + y^2 + z^2 - 2\theta z &= 2v, \\ Ax + By + C(z - \theta) &= m, \\ Adx + Bdy + Cdz &= 0; \end{aligned}$$

where θ has been written in place of γ : m is a given constant; a, b, l are functions of t ; v is a function of θ . The equation $ax + by = lm$ evidently denotes that the planes of the plane curves of curvature are all of them parallel to the axis of z , or what is the same thing, they envelope a cylinder; in the particular case $m = 0$, they all of them pass through the axis of z . In the general case, the required surface is the parallel surface, at the normal distance m , to the surface which belongs to the particular case $m = 0$. This is not assumed in the investigation which follows, but it will be readily perceived how the theorem is involved in, and in fact proved by, the investigation.

I obtain the solution synthetically as follows:

Taking T, a, b functions of t , $a^2 + b^2 = 1$; T_1, a_1, b_1 their derived functions, $aa_1 + bb_1 = 0$; $\Omega = \frac{T_1^2}{4T^2} + a_1^2 + b_1^2$; Θ a function of θ , Θ' its derived function,

$M = \frac{2\Theta}{\Theta'}$; $P = \frac{2\sqrt{T\Theta}}{T + \Theta}$, $Q = \frac{T - \Theta}{T + \Theta}$, and therefore $P^2 + Q^2 = 1$; then writing

$$A_0 = \frac{1}{\sqrt{\Omega}} \left(-b \frac{T_1}{2T} + b_1 Q \right),$$

$$B_0 = \frac{1}{\sqrt{\Omega}} \left(a \frac{T_1}{2T} - a_1 Q \right),$$

$$C_0 = \frac{-1}{\sqrt{\Omega}} (ab_1 - a_1b) P,$$

(where $A_0^2 + B_0^2 + C_0^2 = 1$) we assume

$$x = mA_0 + aMP,$$

$$y = mB_0 + bMP,$$

$$z = mC_0 + \theta + MQ,$$

equations which determine x, y, z as functions of the parameters t and θ . As will presently be shown, we have $A_0 dx + B_0 dy + C_0 dz = 0$; and we have thus $A, B, C = A_0, B_0, C_0$; and this being so, we easily verify the six equations

$$A^2 + B^2 + C^2 = 1,$$

$$bx - ay = m \left(-\frac{T_1}{2T} \frac{1}{\sqrt{\Omega}} \right),$$

$$bA - aB = \left(-\frac{T_1}{2T} \frac{1}{\sqrt{\Omega}} \right),$$

$$x^2 + y^2 + (z - \theta)^2 = m^2 + M^2 + \theta^2,$$

$$Ax + By + C(z - \theta) = m,$$

$$A dx + B dy + C dz = 0,$$

which are the six equations of the problem with the values $a = b, b = -a, l = -\frac{2T_1}{T} \frac{1}{\sqrt{\Omega}}, 2v = m^2 + M^2$, for a, b, l and $2v$.

We in fact at once obtain the third equation $bA_0 - aB_0 = -\frac{T_1}{2T} \frac{1}{\sqrt{\Omega}}$, and

thence the second equation $bx - ay = m(bA_0 - aB_0) = m \left(-\frac{T_1}{2T} \frac{1}{\sqrt{\Omega}} \right)$; then for the fifth equation we have

$$A_0 x + B_0 y + C_0 (z - \theta) = m + M \{ (A_0 a + B_0 b) P + C_0 Q \} = m,$$

since $(A_0 a + B_0 b) P + C_0 Q = 0$; and for the fourth equation we have

$$x^2 + y^2 + (z - \theta)^2 = m^2 + 2m \{ M(A_0 a + B_0 b) P + C_0 Q \} + M^2 = m^2 + M^2.$$

It remains only to prove the assumed equation $A_0 dx + B_0 dy + C_0 dz = 0$. Writing for a moment $X, Y, Z = aMP, bMP, \theta + MQ$ we have

$$\begin{aligned} A_0 dx + B_0 dy + C_0 dz &= A_0 (mdA_0 + dX) + B_0 (mdB_0 + dY) + C_0 (mdC_0 + dZ), \\ &= A_0 dX + B_0 dY + C_0 dZ, \end{aligned}$$

since $A_0 dA_0 + B_0 dB_0 + C_0 dC_0 = 0$ in virtue of $A_0^2 + B_0^2 + C_0^2 = 1$.

We have thus to show that if $X, Y, Z = aMP, bMP, \theta + MQ$, then $A_0 dX + B_0 dY + C_0 dZ = 0$; say we have

$$\begin{aligned} dX &= p dt + p' d\theta, \\ dY &= q dt + q' d\theta, \\ dZ &= r dt + r' d\theta, \end{aligned}$$

then the required values of A_0, B_0, C_0 are proportional to $qr' - q'r, rp' - r'p, pq' - p'q$, and the sum of their squares is $= 1$. Writing for shortness $MP = R, MQ = S$, we have

$$\begin{aligned} p &= a_1 R + a R_1, & p' &= a R', \\ q &= b_1 R + b R_1, & q' &= b R', \\ r &= S_1, & r' &= 1 + S'; \end{aligned}$$

hence

$$\begin{aligned} qr' - q'r &= b [R_1 (1 + S') - R' S_1] + b_1 R (1 + S'), \\ rp' - r'p &= -a [R_1 (1 + S') - R' S_1] - a_1 R (1 + S'), \\ pq' - p'q &= - (ab_1 - a_1 b) R R'. \end{aligned}$$

Here

$$\begin{aligned} R' &= MP' + M'P, & R_1 &= MP_1, \\ S' &= MQ' + M'Q, & S_1 &= MQ_1, \end{aligned}$$

and hence

$$\begin{aligned} RS' - R'S &= M^2 (PQ' - P'Q), \\ R_1 S' - R'S_1 &= M^2 (P_1 Q' - P' Q_1) + MM' (P_1 Q - P Q_1); \end{aligned}$$

moreover, from the values of P and Q we have

$$\begin{aligned} P' &= \frac{\Theta'}{2\Theta} PQ, & P_1 &= -\frac{T_1}{2T} PQ, \\ Q' &= -\frac{\Theta'}{2\Theta} P^2 &= -\frac{P^2}{M}, & Q_1 &= \frac{T_1}{2T} P^2, \end{aligned}$$

and thence

$$P_1 Q - P Q_1 = -\frac{T_1 P}{2T}; \quad P Q' - P' Q = -\frac{P}{M}; \quad P_1 Q' - P' Q_1 = 0;$$

also

$$\begin{aligned} R' &= PQ + M'P, = P(Q + M'), & 1 + S &= 1 - P^2 + M'Q, = Q(Q + M'); \\ R_1 &= -\frac{T_1}{2T} MPQ, & RS' - R'S &= -PM, & R_1S' - R'S_1 &= -\frac{T_1}{2T} MPM', \end{aligned}$$

and consequently

$$\begin{aligned} R_1(1 + S') - R'S_1 &= -\frac{T_1}{2T} MP(Q + M'), \\ R(1 + S') &= MPQ(Q + M'), \\ RR' &= MP^2(Q + M'), \end{aligned}$$

and hence the foregoing expressions for $qr' - q'r$, $rp' - r'p$, $pq' - p'q$ each contain the factor $MP(Q + M')$; and, omitting this factor, the expressions are

$$\left\{ -b \frac{T_1}{2T} + b_1Q \right\}, \quad \left\{ a \frac{T_1}{2T} - a_1Q \right\}, \quad -(ab_1 - a_1b)P;$$

the sum of the squares of these values is $= \frac{T_1^2}{4T^2} + a_1^2 + b_1^2 = \Omega$, and we have thus the required values

$$\begin{aligned} A_0 &= \frac{1}{\sqrt{\Omega}} \left\{ -b \frac{T_1}{2T} + b_1Q \right\}, \\ B_0 &= \frac{1}{\sqrt{\Omega}} \left\{ a \frac{T_1}{2T} - a_1Q \right\}, \\ C_0 &= \frac{-1}{\sqrt{\Omega}} (ab_1 - a_1b)P, \end{aligned}$$

which completes the proof.

In the case $m = 0$, the solution is

$$x, y, z = \frac{2\Theta}{\Theta'} a \frac{2\sqrt{T\Theta}}{T + \Theta}, \quad \frac{2\Theta}{\Theta'} b \frac{2\sqrt{T\Theta}}{T + \Theta}, \quad \theta + \frac{2\Theta}{\Theta'} \frac{T - \Theta}{T + \Theta}.$$

Bonnet, in the paper (Jour. Ecole Polyt. t. 20) referred to at the beginning of this memoir, gives for this case (see p. 199) a solution which he says is equivalent to that obtained by Joachimsthal in the paper "Demonstrationes theorematum ad superficies curvas spectantium," Crelle, t. 30 (1846), pp. 347-350; viz. Joachimsthal's form is

$$\begin{aligned} x &= \frac{\mu \sin L \sin \lambda}{1 + \cos L \cos M}, \\ y &= \frac{\mu \sin L \cos \lambda}{1 + \cos L \cos M}, \\ z &= \frac{\mu \cos L \sin M}{1 + \cos L \cos M} + \int \cot M d\mu, \end{aligned}$$

where L, M denote arbitrary functions of the parameters λ, μ respectively. To identify these with the foregoing form, I write

$$\begin{aligned}\sin \lambda &= -a, \cos L = -\frac{T-1}{T+1}, \cos M = \frac{\Theta-1}{\Theta+1}, \mu = \frac{4\Theta\sqrt{\Theta}}{\Theta'(\Theta+1)}; \\ \cos \lambda &= -b, \sin L = \frac{-2\sqrt{T}}{T+1}, \sin M = \frac{-2\sqrt{\Theta}}{\Theta+1};\end{aligned}$$

we thus have

$$\frac{\sin L \sin \lambda}{1 + \cos L \cos M} = a \frac{2\sqrt{T}}{T+1} \div 1 - \frac{(T-1)(\Theta-1)}{(T+1)(\Theta+1)}, = \frac{a\sqrt{T}(\Theta+1)}{T+\Theta},$$

and thence

$$x = \frac{2\Theta}{\Theta'} a \frac{2\sqrt{T\Theta}}{T+\Theta}, \quad y = \frac{2\Theta}{\Theta'} b \frac{2\sqrt{T\Theta}}{T+\Theta}.$$

Moreover,

$$\frac{\cos L \sin M}{1 + \cos L \cos M} = \frac{\sqrt{\Theta}(T-1)}{T+\Theta},$$

and thence the first term of z is $= \frac{2\Theta}{\Theta'} \frac{2\Theta(T-1)}{(\Theta+1)(T+\Theta)}$; or observing that $2\Theta(T-1) = (\Theta+1)(T-\Theta) + (\Theta-1)(T+\Theta)$, this is

$$= \frac{2\Theta}{\Theta'} \frac{T-\Theta}{T+\Theta} + \frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)},$$

or we have

$$z = \frac{2\Theta}{\Theta'} \frac{T-\Theta}{T+\Theta} + \frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)} + \int \cot M d\mu.$$

Here

$$\begin{aligned}\cot M d\mu &= -\frac{\Theta-1}{2\sqrt{\Theta}} \frac{4\Theta\sqrt{\Theta}}{\Theta(\Theta+1)} \left\{ \frac{\frac{3}{2}\Theta'}{\Theta} - \frac{\Theta'}{\Theta+1} - \frac{\Theta''}{\Theta'} \right\} d\theta \\ &= \frac{\Theta(\Theta-1)}{\Theta+1} \left\{ -\frac{3}{\Theta} + \frac{2}{\Theta+1} + \frac{2\Theta''}{\Theta'^2} \right\} d\theta.\end{aligned}$$

But writing

$$\xi = \frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)},$$

we have

$$\begin{aligned}d\xi &= \frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)} \left\{ \frac{\Theta'}{\Theta} + \frac{\Theta'}{\Theta-1} - \frac{\Theta'}{\Theta+1} - \frac{\Theta''}{\Theta'} \right\} d\theta \\ &= \frac{\Theta(\Theta-1)}{\Theta+1} \left\{ \frac{2}{\Theta} + \frac{2}{\Theta-1} - \frac{2}{\Theta+1} - \frac{2\Theta''}{\Theta'^2} \right\} d\theta,\end{aligned}$$

and thence

$$d\xi + \cot M d\mu = \frac{\Theta(\Theta - 1)}{\Theta + 1} \left\{ -\frac{1}{\Theta} + \frac{2}{\Theta - 1} \right\} d\theta, = d\theta,$$

and consequently

$$\xi + \int \cot M d\mu = \theta,$$

and the value of z thus is

$$z = \theta + \frac{2\Theta}{\Theta'} \frac{T - \Theta}{T + \Theta},$$

which completes the identification.

Bonnet's formulae just referred to, making a slight change of notation and correcting a sign, are

$$\begin{aligned} x &= \frac{\Gamma' \sin \theta}{\cos i(c + \Theta)}, \\ y &= \frac{\Gamma' \cos \theta}{\cos i(c + \Theta)}, \\ z &= \Gamma + i\Gamma' \tan i(c + \Theta), \end{aligned}$$

where Γ, Θ are arbitrary functions of the parameters c, θ respectively. To identify these with Joachimsthal's, write

$$\begin{aligned} \sin \lambda &= \sin \theta, & \cos M &= i \cot ic, & \cos L &= i \cot i\Theta, & \mu &= -\Gamma' \operatorname{cosec} ic, \\ \cos \lambda &= \cos \theta, & \sin M &= \operatorname{cosec} ic, & \sin L &= \operatorname{cosec} i\Theta, \\ & & \cot M &= i \cos ic, & \cot L &= i \cos i\Theta, \end{aligned}$$

we have

$$x = \frac{\mu \operatorname{cosec} i\Theta \sin \theta}{1 - \cot ic \cot i\Theta} = \frac{-\mu \sin ic \sin \theta}{\cos i(c + \Theta)}, = \frac{\Gamma' \sin \theta}{\cos i(c + \Theta)};$$

and similarly

$$y = \frac{\Gamma' \cos \theta}{\cos i(c + \Theta)}.$$

Moreover, the first term of z is

$$\frac{\mu i \cot i\Theta \operatorname{cosec} ic}{1 - \cot ic \cot i\Theta} = \frac{-\mu i \cos i\Theta}{\cos i(c + \Theta)} = \frac{i\Gamma' \cos i\Theta}{\sin ic \cos i(c + \Theta)};$$

or since $i\Theta = i(c + \Theta) - ic$, and thence

$$\begin{aligned} \cos i\Theta &= \cos ic \cos i(c + \Theta) + \sin ic \sin i(c + \Theta), \\ &= i\Gamma' \{ \cot ic + \tan i(c + \Theta) \}, \end{aligned}$$

this is

and we have

$$z = i\Gamma' \tan i(c + \Theta) + i\Gamma' \cot ic + \int \cot M d\mu.$$

But from the equation $\mu = -\Gamma' \operatorname{cosec} ic$, we obtain

$$d\mu = (-\Gamma'' \operatorname{cosec} ic + i\Gamma' \operatorname{cosec} ic \cot ic) dc;$$

whence $\cot M d\mu = (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic) dc$,

and thence

$$d(i\Gamma' \cot ic + \int \cot M d\mu) = (i\Gamma'' \cot ic + \Gamma' \operatorname{cosec}^2 ic) dc \\ + (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic) dc, = \Gamma' dc;$$

that is

$$i\Gamma' \cot ic + \int \cot M d\mu = \Gamma,$$

and consequently

$$z = \Gamma + i\Gamma' \tan i(c + \Theta),$$

which completes the identification of Bonnet's formula with Joachimsthal's.

SS. THE SETS OF CURVES OF CURVES OF CURVATURE EACH SPHERICAL.

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ x^2 + y^2 + z^2 - 2ax - 2by - 2cz - 2u &= 0, \\ A(x-a) + B(y-b) + C(z-c) - l &= 0, \\ x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z - 2v &= 0, \\ A(x-\alpha) + B(y-\beta) + C(z-\gamma) - \lambda &= 0, \\ Adx + Bdy + Cdz &= 0; \end{aligned}$$

the condition being

$$a\alpha + b\beta + c\gamma - l\lambda + u + v = 0.$$

The cases are

	a	b	c	l	u	α	β	γ	λ	v
$SS1^0$	0	0	0	l	$\frac{1}{2}(ml + m')$	α	β	γ	$\frac{1}{2}m$	$-\frac{1}{2}m'$
$SS2^0$	0	0	c	l	$\frac{1}{2}(ml + m')$	α	β	0	$\frac{1}{2}m$	$-\frac{1}{2}m'$
$SS3^0$	0	0	c	$mc + \frac{1}{2}m'$	$-\frac{1}{2}m''c - m'''$	α	β	$m\lambda + \frac{1}{2}m''$	λ	$\frac{1}{2}m'\lambda + m'''$
$SS4^0$	0	b	c	$mc + m'$	$mm''c + m'm'' - m'''$	α	0	γ	$\frac{1}{m}\gamma + m''$	$\frac{m'}{m}\gamma + m'''$

where m, m', m'', m''' are constants; b, c, l functions of t ; $\alpha, \beta, \gamma, \lambda$ functions of θ .

$SS1^0$ gives circles (i. e. the curves of curvature of one set are circles).

$SS2^0$ is Serret's first case of SS .

$SS3^0$ gives circles.

$SS4^0$ is Serret's second case of SS .

$SS^2 = \text{SERRET'S FIRST CASE OF } SS.$

Writing for convenience $m' = -f^2$, the six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ x^2 + y^2 + z^2 - 2cz - ml + f^2 &= 0, \\ Ax + By + C(z - c) - l &= 0, \\ x^2 + y^2 + z^2 - 2ax - 2\beta y - f^2 &= 0, \\ A(x - \alpha) + B(y - \beta) + C(z - \lambda) &= 0, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

where m, f are constants; c, l are functions of t ; α, β, λ functions of θ . The first set of spheres have no points in common, but the second set have in common the two points $x = 0, y = 0, z = \pm f$. Hence inverting (by reciprocal radius vectors) with one of these points, say $(0, 0, f)$ as centre, the spheres of the first set will continue spheres, but the spheres of the second set will be changed into planes, and the required surface is thus the inversion of a surface PS , which is in fact PS^3 : say this surface PS is the "Inversion" of SS . We invert by the formulae

$$x = \frac{K^2 X}{\Omega}, \quad y = \frac{K^2 Y}{\Omega}, \quad z - f = \frac{K^2 (Z - f)}{\Omega},$$

where $\Omega = X^2 + Y^2 + (Z - f)^2$.

Writing the equation for the second set of spheres in the form

$$x^2 + y^2 + (z - f)^2 - 2ax - 2\beta y + 2f(z - f) = 0,$$

the transformed equation is at once found to be

$$- 2\alpha X - 2\beta Y + 2f(Z - f) + K^2 = 0,$$

or say

$$\alpha X + \beta Y - fZ + f^2 - \frac{1}{2} K^2 = 0;$$

viz. this gives the planes of the Inversion.

Similarly for the first set of spheres, writing the equation in the form

$$x^2 + y^2 + (z - f)^2 + 2(f - c)(z - f) + 2f(f - c) - ml = 0,$$

the transformed equation is found to be

$$\{2f(f - c) - ml\}\{X^2 + Y^2 + (Z - f)^2\} + 2(f - c)K^2(Z - f) + K^4 = 0;$$

viz. this is

$$\begin{aligned} \{2f(f - c) - ml\}(X^2 + Y^2 + Z^2) + 2Z\{(f - c)(-2f^2 + K^2) + fml\} \\ + \{2f(f - c)(f^2 - K^2) - f^2 ml + K^4\} = 0, \end{aligned}$$

which gives the spheres of the Inversion: the two equations take a more simple form if we write therein $K^2 = 2f^2$; viz. they then become

$$\begin{aligned} \alpha X + \beta Y - fZ &= 0, \\ (2f^2 - 2fc - ml)(X^2 + Y^2 + Z^2) + 2Zfml + f^2(2f^2 + 2cf - ml) &= 0; \end{aligned}$$

or say these are

$$\begin{cases} \frac{\alpha X + \beta Y - fZ}{\sqrt{\alpha^2 + \beta^2 + f^2}} = 0, \\ X^2 + Y^2 + Z^2 + \frac{2fml}{2f^2 - 2fc - ml} Z + \frac{f^2(2f^2 + 2cf - ml)}{2f^2 - 2fc - ml} = 0. \end{cases}$$

Interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$\begin{aligned} \alpha X + bY + cZ &= 0 \quad (\text{where } a^2 + b^2 + c^2 = 1), \\ X^2 + Y^2 + Z^2 - 2\gamma Z - 2v &= 0, \end{aligned}$$

and the Inversion is thus a surface $PS3^0$.

$SS4^0 = \text{SERRET'S SECOND CASE OF } SS.$

Writing for convenience $m' = -mf$, $m''' = \frac{1}{2}(e^2 + f^2)$, $mm'' = -g$, and therefore $m'm'' = fg$, the six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ x^2 + y^2 + z^2 - 2by - 2c(z - g) - 2fg + e^2 + f^2 &= 0, \\ Ax + B(y - b) + C(z - c) + m(f - c) &= 0, \\ x^2 + y^2 + z^2 - 2ax - 2\gamma(z - f) - e^2 - f^2 &= 0, \\ A(x - \alpha) + By + C(z - \gamma) - \frac{1}{m}(g + \gamma) &= 0, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

where e, f, g, m are constants; b, c are functions of t ; α, γ functions of θ .

The spheres of the first set pass all of them through the two points

$$x = \pm \sqrt{2fg - e^2 - f^2 - g^2}, \quad y = 0, \quad z = g,$$

and those of the second set pass all of them through the two points

$$x' = 0, \quad y' = \pm e, \quad z' = f,$$

where observe that these are such that

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = 0;$$

viz. the distance of each point of the first pair from each point of the second pair is $= 0$. The pairs of points are one real, the other imaginary, but this is quite consistent with the reality of the spheres.

The first pair of points lie in a line parallel to the axis of x , meeting the axis of z at the point $z = g$; and the second pair in a line parallel to the axis of y , cutting the axis of z at the point $z = f$. It is clear that we can, without loss of generality, by moving the origin along the axis of z , in effect make g to be $= -f$; the equations of the two sets of spheres thus become

$$\begin{aligned}x^2 + y^2 + z^2 - 2by - 2c(z + f) + e^2 + 3f^2 &= 0, \\x^2 + y^2 + z^2 - 2ax - 2\gamma(z - f) - e^2 - f^2 &= 0,\end{aligned}$$

or, if in these equations for e^2 we write $e^2 - 2f^2$, the equations become

$$\begin{aligned}x^2 + y^2 + z^2 - 2by - 2c(z + f) + e^2 + f^2 &= 0, \\x^2 + y^2 + z^2 - 2ax - 2\gamma(z - f) - e^2 + f^2 &= 0,\end{aligned}$$

which are very symmetrical forms.

The spheres of the first set pass through the two points

$$\pm \sqrt{-e^2 - 2f^2}, \quad 0, \quad -f,$$

and those of the second set through the two points

$$0, \quad \pm \sqrt{e^2 - 2f^2}, \quad f,$$

where, of course, the two pairs of points are related as is mentioned above.

By taking as centre of inversion a point of the first pair, we invert the first set of spheres into planes and the second set into spheres; and similarly, by taking a point of the second pair, we invert the first set of spheres into spheres and the second set into planes. By reason of the symmetry of the system it is quite indifferent which point is chosen; and taking it to be a point of the second pair, and writing for convenience $n = \sqrt{e^2 - 2f^2}$ (n is in fact the quantity originally denoted by e), then the points of the first pair are

$$\begin{aligned}\pm \sqrt{-n^2 - 4f^2}, \quad 0, \quad -f, \\0, \quad \pm n, \quad f,\end{aligned}$$

and I take for centre of inversion the point $(0, n, f)$.

Observe that if $e = 0$, $f = 0$, then the four points coincide at the origin, and taking this as centre of inversion, the two sets of spheres are each changed into planes, and the Inversion of the surface SS is thus a surface PP ; this particular case will be considered further on, but I first consider the general case.

The formulae of inversion are

$$x = \frac{K^2 X}{\Omega}, \quad y - n = \frac{K^2 (Y - n)}{\Omega}, \quad z - f = \frac{K^2 (Z - f)}{\Omega},$$

where $\Omega = X^2 + (Y - n)^2 + (Z - f)^2$.

Writing the equation of the second set of spheres in the form

$$x^2 + (y - n)^2 + (z - f)^2 - 2ax + 2n(y - n) + 2(f - \gamma)(z - f) = 0,$$

the transformed equation is

$$-\alpha X + n(Y - n) + (f - \gamma)(Z - f) + \frac{1}{2} K^2 = 0,$$

which gives the planes of the Inversion.

Similarly writing the equation of the first set of spheres in the form

$$x^2 + (y - n)^2 + (z - f)^2 + 2(n - b)(y - n) + 2(f - c)(z - f) + 2n^2 - 2bn + 4f(f - c) = 0,$$

the transformed equation is

$$\{n^2 - bn + 2f(f - c)\} \{X^2 + (Y - n)^2 + (Z - f)^2\} + K^2 \{(n - b)(Y - n) + (f - c)(Z - f)\} + \frac{1}{2} K^4 = 0,$$

which gives the spheres of the Inversion.

Changing the origin, the two equations may be written

$$\begin{aligned} -\alpha X + nY + (f - \gamma)Z + \frac{1}{2} K^2 &= 0, \\ \{n^2 - bn + 2f(f - c)\}(X^2 + Y^2 + Z^2) + K^2 \{(n - b)Y + (f - c)Z\} + \frac{1}{2} K^4 &= 0. \end{aligned}$$

I stop to consider a particular case: Suppose $n = 0$, the equations are

$$\begin{aligned} -\alpha X + (f - \gamma)Z + \frac{1}{2} K^2 &= 0, \\ X^2 + Y^2 + Z^2 - \frac{bK^2}{2f(f - c)}Y + \frac{K^2}{2f}Z + \frac{K^4}{4f(f - c)} &= 0, \end{aligned}$$

or, interchanging herein Y and Z , they are

$$\begin{aligned} -\alpha X + (f - \gamma)Y + \frac{1}{2} K^2 &= 0, \\ X^2 + Y^2 + Z^2 + \frac{K^2}{2f}Y - \frac{bK^2}{2f(f - c)}Z + \frac{K^4}{4f(f - c)} &= 0, \end{aligned}$$

and if for Y we write $Y - \frac{K^2}{4f}$, then the equations become

$$\begin{aligned} -\alpha X + (f - \gamma)Y + K^2 \frac{f + \gamma}{4f} &= 0, \\ X^2 + Y^2 + Z^2 - \frac{bK^2}{2f(f - c)}Z + \frac{K^4(5f - 4c)}{f^2(f - c)} &= 0, \end{aligned}$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$\begin{aligned} aX + bY &= lm, \\ X^2 + Y^2 + Z^2 - 2\theta Z &= 2v, \end{aligned}$$

which belong to $PS4^0$. Hence in this particular case, $n=0$, the Inversion is $PS4^0$.

Reverting to the general case, and to the two equations obtained above, observe that in the second of the two equations the terms in Y, Z have the variable coefficients $n-b$ and $f-c$; so that it does not at first sight seem as if these terms could be by a transformation of coordinates reduced to a single term.

But if again changing the origin we write $Y - \frac{\frac{1}{2}K^2}{n}$ for Y , the two equations become

$$\begin{aligned} -\alpha X + nY + (f-\gamma)Z &= 0, \\ \{n^2 - bn + 2f(f-c)\}\{X^2 + Y^2 + Z^2\} + \frac{K^2}{n}(f-c)(-2fY + nZ) \\ &\quad + \frac{K^4}{4n^2}\{n^2 + bn + 2f(f-c)\} = 0, \end{aligned}$$

where, in the second equation, the terms in Y, Z present themselves in the combination $-2fY + nZ$ with the constant coefficients $-2f$ and n . Hence writing

$$\begin{aligned} \sqrt{n^2 + 4f^2}Y &= nY' - 2fZ', \\ \sqrt{n^2 + 4f^2}Z &= 2fY' + nZ', \end{aligned}$$

and consequently $-2fY + nZ = \sqrt{n^2 + 4f^2}Z'$, and (after the transformation) removing the accents, the equations become

$$\begin{aligned} -\alpha X + \frac{1}{\sqrt{n^2 + 4f^2}}[\{n^2 + 2f(f-\gamma)\}Y - n(f+\gamma)Z] &= 0, \\ X^2 + Y^2 + Z^2 + \frac{K^2(f-c)\sqrt{n^2 + 4f^2}}{n\{n^2 - bn + 2f(f-c)\}}Z + \frac{K^4\{n^2 + bn + 2f(f-c)\}}{n^2\{n^2 - bn + 2f(f-c)\}} &= 0, \end{aligned}$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$\begin{aligned} aX + bY + cZ &= 0, \\ X^2 + Y^2 + Z^2 - 2m\phi Z &= \theta, \end{aligned}$$

which belong to the case $PS3^0$. Hence in this general case the Inversion is a surface $PS3^0$.

I have spoken above of the particular case $e = 0, f = 0$: here the equations of the two sets of spheres are

$$\begin{aligned}x^2 + y^2 + z^2 - 2by - 2cz &= 0, \\x^2 + y^2 + z^2 - 2ax - 2\gamma z &= 0,\end{aligned}$$

which have the origin as a common point. Taking this as the centre of inversion, or writing

$$x = \frac{K^2 X}{\Omega}, \quad y = \frac{K^2 Y}{\Omega}, \quad z = \frac{K^2 Z}{\Omega}, \quad \text{where } \Omega = X^2 + Y^2 + Z^2,$$

the transformed equations are

$$\begin{aligned}bY + cZ - \frac{1}{2}K^2 &= 0, \\ \alpha X + \gamma Z - \frac{1}{2}K^2 &= 0,\end{aligned}$$

or, interchanging X and Y , say

$$\begin{aligned}bX + cZ - \frac{1}{2}K^2 &= 0, \\ \alpha Y + \gamma Z - \frac{1}{2}K^2 &= 0,\end{aligned}$$

which are of the form

$$\begin{aligned}X + tZ - P &= 0, \\ Y + \theta Z - \Pi &= 0,\end{aligned}$$

belonging to a surface $PP3^0$. Hence in this case the Inversion is a surface $PP3^0$.

It thus appears that the surface $SS4^0$ has an Inversion which is either $PS3^0$, $PS4^0$ or $PP3^0$. The inversion has in some cases to be performed in regard to an imaginary centre of inversion.

It was previously shown that the surface $SS3^0$ had an Inversion $PS3^0$, and we thus arrive at the conclusion that a surface SS , with its two sets of curves of curvature each spherical, is in every case the Inversion of a surface PS with one set plane and the other spherical, or else of a surface PP with each set plane. Serret notices that the centre of inversion may be imaginary: this (he says) presents no difficulty, but he adds that it is easy to see that the centres of inversion may be taken to be real, provided that we join to the surfaces thus obtained all the parallel surfaces.

It seems to me that there is room for further investigation as to the surfaces SS : first, without employing the theory of inversion, it would be desirable to obtain the several forms by direct integration, as was done in regard to the surfaces PP and PS ; secondly, starting from the several surfaces PP and PS considered as known forms, it would be desirable to obtain from these, by inver-

sion in regard to an arbitrary centre, or with regard to a centre in any special position, the several forms of the surfaces SS . But I do not at present propose to consider either of these questions.

In conclusion, I remark that I have throughout assumed Serret's *negative* conclusions, viz. that the several cases, other than those considered in the present memoir, give only developable surfaces, or else surfaces having for one set of their curves of curvature circles. These being excluded from consideration, there remain

PP , Serret's two cases $PP1^0$, $PP3^0$;
 PS , his three cases $PS1^0$, $PS3^0$, $PS4^0$;
 SS , his two cases $SS2^0$ and $SS4^0$;

but $PP1^0$ is a particular case of, and so may be included in, $PP3^0$; and similarly $PS1^0$ is a particular case of, and may be included in, $PS3^0$; the cases considered thus are

$PP3^0$; $PS3^0$, $PS4^0$; $SS2^0$ and $SS4^0$.

It would however appear by what precedes that the case $SS4^0$ includes several cases which it is possible might properly be regarded as distinct; and the classification of the surfaces SS can hardly be considered satisfactory; it would seem that there should be at any rate 3 cases, viz. the surfaces which are the Inversions of $PP3^0$, $PS3^0$ and $PS4^0$ respectively.

I regard the present memoir as a development of the analytical theory of the surfaces $PP3^0$, $PS3^0$ and $PS4^0$.